REVIEW

The Ghost of Stochastic Resonance

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Nonlinear systems driven by noise and periodic forces with more than one frequency exhibit the phenomena of Ghost Stochastic Resonance (GSR). This has been found in a wide and disparate variety of fields ranging from biology to geophysics where the common feature is the emergence of a “ghost” frequency in the output which is absent in the input. As reviewed here, the uncovering of this phenomena helped to understand a range of problems, from the perception of pitch in complex sounds or visual stimuli, to the explanation of climate cycles. Recent theoretical efforts show that a simple mechanism with two ingredients are at work in all these observations. The first one is the linear interference of the periodic inputs and the second a nonlinear detection of the largest constructive interferences, involving a noisy threshold. These notes are dedicated to review the main aspects of this phenomenon, as well as its different manifestations described on a bewildering variety of systems ranging from neurons, semiconductor lasers, electronic circuits to models of glacial climate cycles.

**Keywords:** Stochastic resonance, complex inharmonic forcing, noise, threshold devices

1. Introduction

1.1. Beyond Stochastic Resonance.

For the last two decades, the study of nonlinear systems driven by noise and periodic signals lead to the uncovering of a phenomenon known as stochastic resonance (SR) \cite{1–7}, in which an optimal level of noise makes the nonlinear system to reproduce more accurately the periodicity of the input signal.

A variant of SR reviewed here is the Ghost Stochastic Resonance (GSR), phenomenon which is ubiquitous for nonlinear systems driven by noise and periodic signals with more than one frequency. GSR was first proposed \cite{8, 9} to explain how a simple neuron suffices to detect the periodicities of complex sounds. Later, similar dynamics was proposed as a mechanism for the quasi-periodicity observed in abrupt temperature change occurrences during the last ice age, known as Dansgaard-Oeschger (DO) events. Between these two examples, other manifestations of GSR were identified in disparate systems including neurons, semiconductors lasers, electronic circuits, visual stimuli, etc. In these notes GSR will be reviewed providing its main underlying ingredients and the most relevant manifestation in different natural phenomena.

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1.2. Two examples of unexplained periodicities.

To introduce the main aspects of GSR, we will use two examples where a system responds with spectral components which are not present in their inputs.

1.2.1. The missing fundamental illusion in pitch perception

Pitch is a subjective attribute by which any sound can be ordered in a linear scale from low to high. If a tone is composed by a single frequency, the perceived pitch is its frequency. In the case of complex sounds, composed of several pure tones, there is no objective measure of the pitch, despite the fact that the accuracy of different observers can be as small as a few percents, even for untrained ears.

A well known illusion used to study pitch perception takes place when two tones of different frequencies are heard together. The paradox is that, under these conditions, the perception corresponds to a third lower pitched tone and not to the two frequencies. This is referred to as the missing fundamental illusion because the perceived pitch corresponds to a fundamental frequency for which there is no actual source of air vibration. A characteristic phenomenon in pitch perception is the so called pitch shift. This refers to the variation of the perceived pitch when the frequencies of an harmonic tone are rigidly displaced. The first quantitative measurements of this phenomena are reproduced in Fig.1 ([10] and [11]).

The experiment is described as follows: A sinusoidal amplitude modulated sound:

\[
s(t) = (1 + \cos(2\pi gt))\sin(2\pi ft) = \frac{1}{2}\sin(2\pi(f - g)t) + \sin(2\pi ft) + \frac{1}{2}\sin(2\pi(f + g)t),
\]

(1)

is presented to a group of subjects which were asked to report the pitch perceived. This is a complex tone composed by three components equispaced in frequency by a value \(g\). If \(f = ng\) with \(n\) integer both three tones are higher harmonics of the fundamental \(g\). If the three components are displaced by \(\Delta f\), e.g, \((f = ng + \Delta f)\), the three tones are no longer higher harmonics of \(g\) \((g\) is not the missing fundamental) but the difference between them remains equal to \(g\). These stimuli are represented in the top diagram of Fig.1 for \(g = 200\) Hz, \(f = 1.4\) kHz and \(\Delta f\) between 0 and 1 kHz.

It was assumed for many years that, under these experimental conditions, the auditory system would report a pitch corresponding to nonlinear distortions [51]. In other words, pitch perception was related to the difference between the intervening tones. If that were the case, the reported pitch should remains constant (i.e., the red dashed line of bottom panel of Fig.1). Instead, the reported pitch falls along straight lines of slopes close to \(1/n\) (but not exactly). In addition, there is an ambiguity since the same stimulus was perceived to have pitch falling in more than one line. These results suggested that the auditory system is using another characteristic of the signal to estimate the pitch perception.

The discussion presented in section 2.2 will show that key features related to how the pitch is perceived can be explained at once in terms of Ghost Stochastic Resonance.

1.2.2. The Dansgaard-Oeschger events

Many paleoclimatic records from the North Atlantic region and Eurasia show a pattern of rapid climate oscillations, the so-called Dansgaard-Oeschger events [12–17], which seem to exhibit a characteristic recurrence time scale of about 1470 years during the second half of the last glacial period [18–20]. Fig. 2 shows the temperature anomalies during the events as reconstructed from the ratio of two stable oxygen isotopes as measured in two deep ice cores from Greenland, the GISP2 (Greenland Ice Sheet Project 2 [13, 60]) and NGRIP (North Greenland Ice Core Project [17]) ice cores, during the interval between 10000 and 42000 years before present (BP). Note that this isotopic ratio is a standard indicator to infer about temperature variations in many paleoclimatic records. The numbers 0 – 10 in Fig. 2 label the Dansgaard-Oeschger events, many of which are almost exactly spaced by intervals of about 1470 years or integer multiples of that
Figure 1. Results from Schouten’s experiments (modified from Ref. [11]) demonstrating that equidistant tones do not produce constant pitch. The authors used a complex sound as described in Eq. 1 with center values from 1.2 to 2.4 kHz. Top diagrams depict three examples of the frequency spectra of the complex sound used (with center frequencies of 1.4, 1.8 and 2.2 kHz). Dotted lines correspond to the missing component $g$. The bottom graph indicates, with symbols (open or filled circles and triangles), the pitch heard by the three subjects for each complex sound. The dashed lines show that a $1/n$ function consistently underestimate the linear relationship between the frequency and pitch shift value.

It has often been hypothesized that solar variability could have played a role in triggering these rapid temperature shifts. However, whereas many solar and solar-terrestrial records, including the historical sunspot record, were reported to exhibit cycles of about 88 and 210 years, no noteworthy spectral component of about 1470 years has been identified in these records [23–29]. A turning point in this discussion is the work of Braun et al. [30], showing that an ocean-atmosphere model can even generate perfectly periodic Dansgaard-Oeschger-like output events, spaced exactly by 1470 years or integer multiples of that value, when driven by input cycles of about 87 and 210 years. In other words, no input power at a spectral component corresponding to 1470 years is needed to generate output events with maximum spectral power at that value. This work indicates that the apparent 1470 year response time of the events could result from a superposition of two shorter input cycles, together with a strong nonlinearity and a millennial relaxation time in the dynamics of the system as another manifestation of the GSR, as will be discussed in section 2.3.

2. Ghost Stochastic Resonance

2.1. A toy model

The two examples described above can be modeled in terms of GSR. Despite some flavors the models are build upon two basic ingredients: linear interference of pure tones plus a threshold, who play the role of a noisy detector of the largest peaks of the input signal. In [8, 9] the nonlinear element is a constant threshold or a neuron’s model. In [31] the nonlinear device is a exponential dependent double-threshold device, as it is shown in Fig. 3.
Figure 2. Greenland temperature anomalies during the time interval between 42,000 and 10,000 years before present (BP) as reconstructed from the ratio of two stable isotopes as measured in two deep ice cores from Greenland, the GISP2 (top) and NGRIP (bottom) ice cores. Numbers label the Dansgaard-Oeschger events 0-10, following standard paleoclimatic convention. Dashed lines are spaced by approximately 1.470 years. Note that many events recur almost exactly in near-multiples of 1.470 years, e.g. the events 0, 1, 2, 3, 5, 7 and 10 as recorded in the GISP2 data. Both ice core record are shown on their respective standard time scale, which was constructed by independent counting of annual layers in the ice cores.

Figure 3. Two examples of GSR toy models. Left panels: Robust nonlinear stochastic detection of the missing fundamental $f_0$ by a constant threshold. (a) Example of a complex sound $S_c$ built by adding two sinusoidal $S_1$ and $S_2$ frequencies: $f_1 = kf_0$ (bottom) and $f_2 = (k+1)f_0$ (middle). Specifically, $x(t) = \frac{1}{4}(A_1\sin(f_12\pi t) + A_2\sin(f_22\pi t))$ with $A_1 = A_2 = 1$; $k = 2$ and $f_0 = 1$Hz in this case. The peaks (asterisks) exhibited by $S_c$ result from the constructive interference of $S_1$ and $S_2$ at the period of the missing fundamental $f_0$. (b) The peaks of the signal shown in (a) can be reliably detected by a noisy threshold, generating inter-spike intervals close to (or to the integer multiples of) the fundamental period. (Replotted from [9]).

Right panels: Conceptual climate model with a non-constant bi-threshold function as described in Braun et al [31]. Panel(c): Complex signal $S_c$ as shown in panel (a) and the bi-threshold exponential function (in red). Panel (d): the same bi-threshold exponential function and the state function (with value 1 if the system is in warm state and 0 if it is in the cold state) for the conceptual climate model.
2.2. Modeling pitch perception

Understanding the mechanism behind pitch perception would lead to understand related issues concerning consonance, music and speech, for example. Many attempts have been made to model pitch perception [33–35], however its neural mechanisms are still controversial [36–41]. In this context, a simple model [8, 9], based on quantifiable and physiologically plausible neural mechanism was proposed. This model is able to interpret in a very simple fashion, key experiments related to the paradox of missing fundamental illusion and pitch shift.

The assumptions of the model are simple. Let consider a complex tone $S_c$ formed by adding pure tones of frequencies $f_1 = k f_0$, $f_2 = (k + 1) f_0$, ..., $f_N = (k + N - 1) f_0$ as an input of a nonlinear threshold device. It can be observed that the harmonic tone $S_c$ exhibits large amplitude peaks (asterisks in left panel of Fig.3) spaced at intervals $T_0 = 1/f_0$. This peaks are the result of a constructive interference between the constitutive tones. The threshold device detects “statistically” (with the help of noise) the largest peaks of $S_c$, which are spaced as the fundamental.

The single neuron model detects only statistically the occurrence of peaks, as not every peak will induce neuronal spikes. The top panels of Fig.4 shows the density distributions of inter-spike intervals $t$ computed as the probability of observing an inter-spike interval of a given fundamental.

From a theoretical point of view, it is expected that the threshold device detects preferentially the highest peaks of $S_c$, produced by the positive interference of the constituent pure tones. If we think in the simplest case of constructive interference of two tones, it arises from the beating phenomenon which results in a carrier frequency of $f^+ = (f_2 + f_1)/2$ modulated in amplitude with a sinusoidal wave of frequency $f^- = (f_2 - f_1)/2$. In this case, the interval between the highest peaks is equal to the nearest integer number $n$ of half-periods of the carrier lying within a half-period of the modulating signal. For the case of two consecutive higher harmonic of a given fundamental $f_0$, $f_1 = k f_0$ and $f_2 = (k + 1) f_0$, it can be obtained that $n = f^+/f^- = 2k + 1$ and the corresponding interval is $n/f^+ = 1/f_0$. If the two frequencies are linearly shifted in $\Delta f$, it is expected that the most probable interval between highest peaks take place at a rate $f_r$ with $1/f_r = n/f^+ = (2k + 1)/(2k + 1)f_0 + 2\Delta f$.

In the general case where $S_c$ is composed by $N$ harmonic tones as described above, the expected resonant frequency $f_r$ should follows the relation:

$$f_r = f_0 + \frac{\Delta f}{k + (N - 1)/2}$$ (2)
Figure 4. Single neuron GSR. Top panels: Density distributions of inter-spike intervals $t$ in the system for three noise intensities. Bottom panel: Signal-to-noise ratio computed as the probability of observing an inter-spike interval of a given $t$ (±5% tolerance) as a function of noise variance $\sigma$ estimated at the two input signals’ time scales: $f_1$ (stars) and $f_2$ (filled circles) as well as for $f_0$ (empty circles). The large resonance is at $f_0$, i.e., a subharmonic which is not present at the input. Reproduced from [8].

Fig. 5 shows results of simulations and theoretical lines predicted from Eq.(2) for $N = 2$ (left panels) and $N = 3$ (right panels) for $k = 1 – 7$. The matching between simulation and theory is remarkable.

If $f_0 = 200$Hz, $N = 3$ and $k = 6$, the stimulus $S_c$ have the same features of those used in [11]. Panel (a) of Fig.6 show the results of these simulations superimposed with psycho-acoustical pitch reports from Schouten et al’s experiments. The results of the simulations are presented as histograms of inter-spike intervals produced by the neuronal device. The experimental results report the pitch detected by the subjects in the experiments. Panel (b) of Fig.6 show the good agreement between the theory described in Eq.(2), simulation data and the pitch estimated from the predominant interspike interval in the discharge patterns of cat auditory nerve fibers in response to complex tones [38, 39]. It should be noticed that, in both cases, the agreement between both is parameter independent.

The simplicity of GSR contrast with previous proposal suggesting complicated mechanisms mediated by relatively sophisticated structures not yet identified as delay lines [36], oscillators in combination with integration circuits [43], neural networks [41], timing nets [34] and others [43, 44]. Finally, it is important to remark that recent experiment in a synthetic cochlea [45] confirms all these results.
Figure 5. Pitch shift simulations for $N = 2$ (Left Panels) and $N = 3$ (Right panels) frequency signals. Top: The probability (as gray scale) of observing a spike with a given instantaneous firing frequency $f_p$ (in the ordinate) as a function of the frequency $f_1$ of the lowest of two components of the input signal (abscissa). The family of lines is the theoretical expectation (Eq.(2)) for $N = 2, 3$ and $k = 1−7$. Bottom: The same data from the top left panel are replotted as input-output frequency ratio vs input frequency $f_1$ ($f_0 = 1\text{Hz}$) ($f_p$ corresponds to $f_r$ in the notation of Eq.2).

Figure 6. Simulations and experiments of pitch shift. Left Panel shows that theoretical prediction of Eq.2 superimpose exactly with the result of simulations (Grey Histograms, $k = 5−9$ and $N = 3$) and reported pitch from Schouten’s experiments (circle and triangle symbols) [11]. Right Panel: Theoretical expectation from Eq.(2) is superimposed here onto the experimental results of Cariani and Delgutte [39]. The physiological pitches estimated from the highest peak of the interspike interval distribution in response to two variable (500–750 Hz) carrier AM tones with modulation frequencies (horizontal dashed lines) of 125 Hz (downward triangles) or 250 Hz (upward triangles) fit the predictions of Eq.2 (diagonal dotted lines with $N = 3$). The agreement in both cases is parameter independent. Reproduced from [9].

2.3. Modeling Dansgaard-Oeschger events

Paleoclimatic records during the last ice age display abrupt changes in temperature which are known as Dansgaard-Oeschberg (DO) events (see Fig.2). Many of these events are almost exactly spaced by about 1470 years or integer multiples, and their trigger and underlying causes are still a matter of debate. One of the most intriguing approaches to understand the mechanisms behind DO events was done in terms of Ghost Stochastic Resonance. In what follows we will show two models which, at different scales and based in GSR, aim to explain the recurrence properties of these warming events.
2.3.1. Ocean-Atmosphere Model

It has been hypothesized [30] that the about 1470-year recurrence time of do events could be caused by two reported solar cycles of centennial scale: the DeVries and Gleissberg cycles with leading spectral components corresponding to periods near 210 and 87 years [23–25, 27, 28] respectively. A carefully inspection of these frequency cycles shows that they are approximately the 7 and 17 harmonic superior of 1470, which leads to the hypothesis that DO events could be caused by some kind of ghost stochastic resonance mechanism. In order to test this hypothesis a coupled Ocean-Atmosphere model (CLIMBER-2) [30] was forced with the two mentioned solar frequencies:

\[ F(t) = -A_1 \cos(2\pi f_1 t + \phi_1) - A_2 \cos(2\pi f_2 t + \phi_2) + K \]  

For simplicity, this forcing was introduced as a variation in freshwater input with \( f_1 = 1/210 \text{ years}^{-1} \) and \( f_2 = 1/86.5 \text{ years}^{-1} \). The amplitudes \( A_1 \) and \( A_2 \) as well as the phases \( \phi_1 \) and \( \phi_2 \) are parameters of the forcing. The constant \( k \) mimics changes in the background climate compared with the Last Glacial Maximum, which is considered as the underlying climate state. Further details of the simulations can be found in [30]. Within a wide range of forcing parameters, this kind of perturbation to the model produces events similar to the DO ones. The simulated events represent transitions between a stadial (“cold”) and interstadial (“warm”) mode of the North Atlantic thermohaline (i.e., heat- and salt-driven) ocean circulation.

In the response of the model three different regimes exist: a “cold regime” in which the thermohaline circulation persist in the stadial mode, a “warm regime” in which the interstadial mode is stable and a “Dansgaard-Oeschberg regime” in which cyclic transition between both modes occur. These transitions result in abrupt warm events in the North Atlantic region, similar to DO events as shown in the results of Fig.7.

Events spaced by 1470 years are found within a continuous range of forcing-parameters. Indeed, this timescale is robust when the phases, the amplitudes and even the frequencies of the two forcing cycles are changed over some range (see [30]). Also noise, when added to the periodic forcing, does not wipe out the preferred tendency of the events to recur almost exactly in integer multiples of 1470 years.

These simulations, which reproduce some of the characteristic recurrence properties of the DO events in the paleoclimatic records, clearly show that a ghost stochastic resonance is at work in the model simulations. Therefore, ghost stochastic resonance is certainly a potential candidate to explain the apparent 1470 years recurrence time scale of Dansgaard-Oeschger events during the last ice age.

2.3.2. Conceptual DO Model

As happened with the analysis of pitch perception in section 2.2, a low dimensional modeling approach can be followed in order to isolate and understand the main dynamical mechanism present in the behavior of the complex model analyzed in last section.

The stability of the simulated 1470 year climate cycle turns out to be a consequence of two well-known properties of the thermohaline circulation: its long characteristic timescale, and the high degree of nonlinearity (that is, the threshold character) inherent in the transitions between the two simulated modes of the thermohaline circulation. A very simple conceptual model that only incorporates these two properties is able to mimic key features of CLIMBER-2, i.e., the existence of three different regimes in the model response, the frequency conversion between forcing and response (that is, the excitation of millennial-scale spectral components in the model response that do not exist in the forcing) and the amplitude dependence of the period in the model response [31]. The general idea of this model is that DO events represent highly nonlinear switches between two different climate states corresponding to the stadial (“cold”) and interstadial (“warm”) modes of the glacial thermohaline circulation.
Figure 7. 1470 years recurrence time scale from the CLIMBER-2 model simulations. All three panels show changes $\Delta T$ in Greenland surface air temperature according to simulations of Braun et al. [30]. The amplitudes of the forcing (Eq.3) are $A_1 = A_2 = 10\,\text{mSv}$. Top, middle and bottom panels correspond to three different values of $K$ ($K = -9\,\text{mSv}$, $-14\,\text{mSv}$ and $-19\,\text{mSv}$) respectively. The dashed lines indicate the position of the global minima in the forcing and are spaced by 1470 years. We can observe that in all cases the events repeat strictly with a period of either 1470 years (bottom) or integer multiples of that value (top, middle). Note that despite the period of four times 1470 years in the middle panel, the average inter-event spacing is 1960 years, and all events in that panel can be divided into three groups, each of which contains only events which recur exactly in integer multiples of 1470 years. Reproduced from [30].

This conceptual model is based in three key assumptions:

1. DO events represent repeated transitions between two different climate states, corresponding to warm and cold condition in the North Atlantic region.
2. These transitions are rapid compared to the characteristic life-time of the two climate states and take place each time a certain threshold is crossed.
3. With the transition between the two states the threshold overshoots and afterwards approaches equilibrium following a millennial-scale relaxation process.

These three assumptions, which are supported by high-resolution paleoclimatic records and/or by simulations with a climate model [31] can be implemented in the following way: A discrete index $s(t)$ that indicates the state of the system at time $t$ (in years) is defined. The two states, warm and cold, correspond with the values $s = 1$ and $s = 0$ respectively. A threshold function $T(t)$ describing the stability of the system at time $t$ is also defined.

The next step is to define the rules for the time evolution of the threshold function $T(t)$. When the system shifts its state, it is assumed that a discontinuity exists in the threshold function: With the switch from the warm state to the cold one (at $t = t'$ in panel (d) of Fig.3) $T$ takes the value $A_0$. Likewise, with the switch from the cold state to the warm one (at $t = t''$ in panel (d) of Fig.3) $T$ takes the value $A_1$. As long as the system does not change its state, the evolution of
$T$ is assumed to be given by a relaxation process

$$T(t) = (A_s - B_s) \cdot \exp\left(-\frac{t - \delta_s}{\tau_s}\right) + B_s,$$

where the index $s$ stands for the current state of the model (i.e., $s=0$ is the warm state and $s=1$ the cold one), $\delta_0$ labels the time of the last switch from the warm state into the cold one and $\delta_1$ labels the time of the last switch from the cold state into the warm one. The third assumption is that the change from one state to another happens when a given forcing function $f(t)$ crosses the threshold function. In the right panel of Fig.3 we plot an schematic representation of the threshold function when the model is forced with a bi-sinusoidal input with frequencies $f_1$ and $f_2$ which are the second and third harmonic of a given $f_0$. As we have seen in previous sections, the peaks of the forcing repeat with a period of $1/f_0$ years, despite the absence of this period in the two forcing series.

A stochastic component represented by white noise of zero mean and amplitude $D$ is added in order to take nonperiodic forcing components into account.

In Fig.8 the response of the model is shown for different values of noise amplitudes. It can be observed that for an optimal noise amplitude the waiting time distribution of the simulated events is peaked at 1470 years. This is reflected in the minimum of the coefficient of variation of the waiting times (i.e., the standard deviation divided by its average) and in the maximum of the signal to noise ratio (calculated as the fraction of same waiting times between events around 1470 years) for optimal values of noise.

![Figure 8. Conceptual model for DO events. Panel (a): Forcing signal plus optimal noise and threshold function $T(t)$. Panel (b): Signal to noise ratio calculated as the fraction of waiting time between events around 10% and 20% of 1470 years (right). Panel (c): Histograms of the inter-event waiting times of the simulated events for optimal noise (left). Panel (d): Coefficient of variation as a function of noise amplitude. Its minimum value indicates the resonance. Parameters: $A_0=-27$, $A_1=27$, $B_0=3$, $B_1=-3$, $\tau_0=1200$, $\tau_1=800$](image-url)
These results demonstrate the occurrence of a GSR mechanism also in this low-dimensional model, which itself was constructed from the dynamics of the events as manifested in the much more complex ocean-atmosphere model CLIMBER-2. Note that the parameter values of the model are modified slightly as compared with the original version of the model [31].

3. Related work

3.1. Binaural pitch perception

Besides the question of how pitch is perceived, another contested debate relates to where perception takes place. Although interval statistics of the neuronal firings [38, 39] show that pitch information is already encoded in peripheral neurons, under other conditions pitch perception can take place at a higher level of neuronal processing [47]. A typical example is found in binaural experiments, in which the two components of a harmonic complex signal enter through different ears. It is known that in that case a (rather weak) low-frequency pitch is perceived. This is called “dichotic pitch”, and can also arise from the binaural interaction between broad-band noises. For example, Cramer and Huggins [48] studied the effect of a dichotic white noise when applying a progressive phase shift across a narrowband of frequencies, centered on 600 Hz, to only one of the channels. With monaural presentation listeners only perceived noise, whereas when using binaural presentation over headphones, listeners perceived a 600Hz tone against a background noise.

Recent work based in GSR show that the binaural effects described above can be explained in the same terms. The model comprises a three neurons structure. Two of them receive one single component of the complex signal, so that each neuron represents detection at a different auditory channel in a binaural presentation, acting upon a third processing neuron [46]. These results showed that the higher-level neuron is able to perceive the pitch, hence providing a neural mechanism for the binaural experiments. The membrane potential of the neurons were simulated via a Morris-Lecar model [49]. The two input neurons were unidirectionally coupled to the response neuron via a synaptic coupling model [50]. Details of the modeling could be found in [46].

Right panel of Fig.9 shows the different stages of the simulation when input neurons are stimulated with pure tones of periods 150 ms and 100 ms respectively. Panels (a) and (b) display the membrane potential with spikes exhibiting the same period of input signals. In panel (c) it can be observed the synaptic current elicited by these two neurons onto the processing neurons. We can observe that global maxima of this current are spaced by 300 ms, i.e., the period of the fundamental frequency which is absent at the input. Finally, in panel (d) the processing neuron produces spike trains of period 300 ms (period of the missing fundamental frequency). These results were simulated without noise and reveals the substrate of the stochastic detection process.

In biological neural networks, each neuron is connected to thousands of neurons whose synaptic connections could be represented as “synaptic noise”. In this configuration this effect is taking into account by adding a white noise term of zero mean and amplitude $D_i$ in the input neurons ($i = 1, 2$) and amplitude $D$ in the processing neuron membrane potential. The firing process of this neurons is then governed by noise. In left panels of Fig.10 we observe the mean time between spikes $\langle T_p \rangle$ (panel (a)), the coefficient of variation ($CV = \sigma_p / \langle T_p \rangle$, panel (b)) and the signal to noise ratio measured as the fraction of pulses spaced around $T_0 = 1/f_0$, $T_1 = 1/f_1$ and $T_2 = 1/f_2$ (panel (c)) as a function of the noise amplitude in the processing neuron, $D$. Right panels show the probability distribution functions of the time between spikes $T_p$ for three values of the noise amplitude $D$: (d) Low (e) optimal and (f) high values. Note the remarkable agreement between these results and those obtained for a single neuron in Fig.4 which confirm the robustness of GSR.

The next step was to check if this model reproduces the pitch shift experiments sketched in
Figure 9. Binaural GSR. Left panel: Schematic representation of binaural mechanism. Right Panel: Deterministic response to a distributed harmonic complex signal. The membrane potential for the three neurons is shown: (a,b) input neurons, (d) processing neuron. The synaptic current acting on processing neuron, is shown in plot (c). The two inputs neurons are stimulated with two sinusoidal signals. Reproduced from [46].

Figure 10. Binaural GSR. Left panels: response of the processing neuron as a function of noise amplitude: (a) mean time between spikes $\langle T_p \rangle$, (b) coefficient of variation, CV, and (c) Signal to noise ratio measured as the fraction of pulses spaced around $T_0 = 1/f_0$, $T_1 = 1/f_1$ and $T_2 = 1/f_2$ as a function of the noise amplitude in the processing neuron, $D$. Right panels: probability distribution functions of the time between spikes $T_p$ for three values of the noise amplitude $D$: (d) Low (e) Optimal, and (f) High values. Reproduced from [46].

Fig. 1 and follows the theoretical predictions of Eq.2. The pure tones driving the input neurons have frequencies $f_1 = k f_0 + \Delta f$ and $f_2 = (k+1) f_0 + \Delta f$. Fig.11 shows the results of the simulation. The instantaneous frequency $f_r = 1/T_p$ follows the straight lines predicted in Eq. 2 for $N = 2$, $k = 2 - 5$ and $f_0 = 1$Hz. It is important to notice that even though the linear superposition of inputs is replaced by a coincidence detection of spikes in this configuration, the preferred frequencies response of the output neuron follows the theoretical predictions made for linear interference between tones. This is consistent with the arguments used to deduce Eq. 2, which looks for the coincidence of maxima of the harmonic tones. Here, the output neuron
detects coincidence of spiking neurons, which also take place preferentially at the maxima of harmonic inputs.

Figure 11. Binaural pitch shift simulations. Probability of observing a spike in the processing neuron with instantaneous rate $f_r$ (in gray scale) as a function of the frequency $f_1$ of one of the input neurons. Note the remarkable agreement of the responses following the lines predicted by Eq. 2 for $k=2,3,4,5$ (dashed lines from top to bottom). Reproduced from ref.[46].

A brain structure candidate for the dynamics of the processing neuron is the inferior colliculus, which receives multiple inputs from a host or more peripheral auditory nuclei. Details of the physiology of this nuclei are still uncertain, but enough evidence suggest that temporal and frequency representation of the inputs are present in the spike timing of their neurons. The results exhibited in this section suggest that the neurons in this nuclei can exhibit the dynamics described here, thus participating in the perception of the binaural pitch. The main consequence of these observations is that pitch information can be extracted mono or binaurally via the same basic principle, i.e., ghost stochastic resonance, operating either at the periphery or at higher sensory levels.

A similar arrangement of three neurons was studied in [52] where two input neurons acts on third processing neuron by means of excitatory connections. The input neurons were described as modification of the Ornstein-Uhlembeck diffusion process [53] where periodic signals of frequencies $f_1 = 2f_0 + \Delta f$ and $f_2 = 3f_0 + \Delta f$ were added to the drift coefficients. The processing neuron was also simulated via a Ornstein-Uhlembeck diffusion process. Simulations in this model confirmed the GSR phenomena in both the harmonic ($\Delta f = 0$) and the inharmonic ($\Delta f \neq 0$) cases [52].

3.2. Beyond pitch: Consonance and Dissonance

Mechanism for the perception and processing of complex signals in auditory system are relevant beyond the identification of pitch. An open challenge in this field is to understand the physiological bases for the phenomena of consonance and dissonance [55–57]. Consonance is usually referred to as the pleasant sound sensation produced by certain combinations of two frequencies played simultaneously. On the other hand, dissonance is the unpleasant sound heard with other frequencies combinations [58]. The oldest theory of consonance and dissonance is due to Pythagoras, who observed that the simpler the ratio between two tones, the more consonant they will be perceived. For example, the consonant octave is characterized by a $1/2$ frequency
ratio between two tones, meanwhile the dissonant semitone is characterized by a $15/16$ ratio. Helmholtz [51] analyzed this phenomena in the more sophisticated scenario of complex tones. When two complex tones are played together, it happen that for some combinations (simple ratio $n/m$) the harmonic frequencies match, while in other cases (more complicated ratios $n/m$) they do not. As the frequency ratio becomes more complicated, the two tones share fewer common harmonics leading to an unpleasant beating sensation or dissonance.

Ushakov et al. [54] used a neuronal configuration similar to the already described in previous sections (two inputs neurons driving a processing neuron as in Fig.9) to investigate the phenomena of consonance and dissonance. In this configuration, the model responds to complex inputs composed by two harmonic signals (with frequencies $f_1$ and $f_2$) generating different sequences of spike trains, depending of the ratio of the inputs frequencies. The authors found that regular patterns of inter-spike interval distribution (ISI) were associated with consonant accords, while dissonant accords produced broader ISI distributions.

![Figure 12. Inter-spike interval distributions of the consonant accords: octave (2/1), perfect fifth (3/2), major third (5/4), and minor third (6/5) for both input neurons ($\rho_1$ and $\rho_2$) and the processing neuron ($\rho_{out}$). The ratio of frequencies ($m/n$) as well as the common musical terminology is denoted for each accord. Reproduced from [54].](image)

Examples in Fig.12 show the ISI distribution of the processing neuron for a group of consonant accords: a octave (2/1), a perfect fifth (3/2), a major third (5/4) and a minor third (6/5). Notice the peaks in the distribution $\rho_{out}$ which are not present in the input patterns of $\rho_1$ and $\rho_2$. Fig.13 show the ISI distributions of the processing neuron for dissonant accords: major second (9/8), minor seventh (16/9), minor second (16/15) and augmented fourth (45/32). The main result here is the fact that ISI distributions are blurred with respect to those obtained for consonant accords, supporting the hypothesis that the perception of consonance in tonal music is associated with more regular firing patterns in response to combinations of harmonic inputs with frequencies $f_2 = (m/n)f_1$.

3.3. A dynamical model for Dansgaard-Oeschger events

In section 2.3 two models were presented in order to explain the characteristic 1470 years observed recurrence time of DO events in terms of a bi-sinusoidal forcing with frequencies close to the main spectral components of two reported solar cycles, the DeVries/Suess and Gleissberg cycles. In this section a model [59] for the evolution of Greenland paleo-temperature during the last
Figure 13. Inter-spike interval distributions of the dissonant accords: major second (9/8), minor seventh (16/9), minor second (16/15), and augmented fourth (45/32). The ratio of frequencies (m/n) as well as the common musical terminology is denoted for each accord. Reproduced from [54]

about 80,000 years (an interval that comprises the last ice age as well as the current warm age called Holocene) is described.

Figure 14. Reconstructed Greenland temperature based on the ratio of two stable oxygen isotopes as measured in the GISP2 deep ice core from Greenland. The time series is divided in three parts: the early ice age, the late ice age and the Holocene. Data from [60].

Fig.14 shows the reconstructed Greenland temperature based on the ratio of two stable oxygen isotopes as measured in the GISP2 deep ice core up to about 80000 years before present. Note that the dating accuracy typically decreases with increasing age of the ice. The temperature-proxy obtained from these records [60] reveals at least three different dynamical regimes: the early ice age, the late ice age and the Holocene. In a simplified approach, these regimes can be interpreted as the result of the switching between two states (a warm and a cold one) driven in
part by noise. The interval between 12000 and 50000 years before present (late ice age) would corresponds to an almost ghost stochastic resonance regime, whereas the last 10000 years (known as holocene) stand for an age without transitions.

The ideas underlying this model are the same as in the model presented in section 2.3: DO events represent highly nonlinear switches between two different climate states corresponding to the glacial cold (i.e., stadial) and the glacial warm (i.e., interstadial) modes of the North Atlantic thermohaline circulation. Note that the Holocene mode of the ocean circulation is assumed to correspond to the glacial interstadial mode. The assumptions in the model can be summarized as follows:

1. The existence of two states, the glacial cold and the glacial warm (or, Holocene) ones.
2. The states represent different modes of operation of the thermohaline circulation in the North Atlantic region.
3. The model is forced by a periodic input with frequencies close to the leading spectral components of the reported De Vries/Suess and Gleissberg cycles (∼207 and ∼87 years respectively) plus a stochastic component.
4. A transition between the states takes place each time a certain threshold is crossed.
5. With every transition the threshold overshoots and afterwards approaches equilibrium following a millennial time scale relaxation process.
6. During the Holocene the periodic forcing is not able by itself to produce transitions, and the climate system remains in the warm state.

The model is presented as a dynamical system submitted to a periodic forcing (with the frequencies mentioned before) in a double well potential and a stochastic component representing non periodic forcing components. It is described with the following set of differential equations:

\[
\dot{x} = \frac{1}{a} \left[ y.(x - x^3) + f(t) + D\sqrt{a}\xi(t) \right] \\
\dot{y} = -\frac{y}{\tau_s} + \delta_s,
\]

where \( y.(x - x^3) = -\frac{dV}{dx} \) and \( V(x) \) is a double well potential with a potential barrier following the dynamics of the \( y \) variable. \( f(t) = F_1(\cos(2\pi f_1 t + \phi_1) + \alpha \sin(2\pi f_2 t + \phi_2)) \) MIMICS the solar forcing with \( f_1 = 207 \text{years}^{-1} \) and \( f_2 = 87 \text{years}^{-1} \). The term \( D\xi(t) \) stands for a white noise process of zero mean and amplitude \( D \) and \( a \) is a scaling constant. In the equation for the threshold dynamics, \( \tau_s \) and \( \delta_s \) were the characteristic time decay and asymptotic threshold respectively (\( s = 1(0) \) for warm (cold) state).

In the absence of noise, this system presents a transition from a low amplitude to a high amplitude oscillations regime. If it is placed in the onset of this transition (i.e., there are no jumps without noise), it is possible to find an optimal amount of noise for which the system switches between warm and cold state every 1470 years approximately, which corresponds to a spectral component that is absent in the periodic input. The system can be placed at different dynamical regimes for different values of the noise amplitude \( D \). Fig.15 shows a comparison between the output of the model \( x \) and the GISP2 reconstructed temperature. The comparison comprises temperatures records as well as their frequency spectra obtained via a Discrete Fourier Transform (DFT). The three dynamical regimes were tuned in order to get the best matching between the Fourier spectra. Even though the comparison between both series is not good enough during the Holocene (panels (a) and (d)), a good agreement exists during the Ice Age where the main features of the dynamics are driven by transitions between the two states. It is important to remark that during the late Ice Age (panels (c) and (f)), the best fit between the temperature record and the simulations correspond to a dynamical scenario very close to the Ghost Stochastic Resonance.
4. Ghost Stochastic Resonance in other systems.

4.1. Visual Stimulation

The robustness of the already discussed models, as well as the absence of fitting parameters, allow to think that GSR could be found in other sensorial systems, like vision, as was suggested in [9]. A phenomenon that shows the analogy between the visual and auditory systems was analyzed in [61], where the subjective flicker rates for compound waveforms without their fundamental frequencies were measured. The questions in this experiments were:

- Could observers perceive flickers at fundamental frequency when it is not present in the stimuli?
- Could this perception be sustained even if higher harmonics are added with random phase conditions?

To answer these questions, eight different stimuli driving a light emitting diode were presented to a group of subjects. All the stimuli were compound waveforms consisting of five components. The frequency of each component corresponded to the $n^{th}$ harmonic of the common $f_0$. These components had equal amplitude and were compounded to make two kind of waveforms for each stimulus: “in-phase” and “random-phase” waveforms for five values of $f_0$ ($f_0 = 0.75 - 3\text{Hz}$). Further details of experimental conditions can be found in [61]. Fig.16 shows the main results of these experiments in which the observers judged the flicker frequency by comparison with flicker of test stimuli with a single frequency. In all cases observers reported to see a flickering rates close to the absent $f_0$ even in the case of random-phase stimuli.
Figure 16. GSR in response to a flickering visual stimuli. Each graph represent the perceived frequency of flickering of a visual stimuli reported by four observers. The abscissa represents the frequency of the comparison stimulus matched with each test stimulus. On the ordinate, the probability of the response is represented in percentiles. Inputs comprised several harmonics (indicated in the figure) with zero (denoted “in-phase”) or random phase differences. Reproduced from ref.[61].

The results of these experiments show that an analogous phenomenon of missing fundamental illusion can be found in spatial vision. Missing fundamental visual patterns were also used in experiments of perception of motion [62], were square-wave gratings without the fundamental were presented to a group of subjects who perceived to move backwards where presented in quarter-cycle jumps (even though their edges and features all move forward). Even tough these experiment were not directly related with GSR, it stress the ubiquity of the missing fundamental phenomena across physiological systems.

4.2. Lasers

The uncovering of GSR phenomena in a variety of environments drove the search of other dynamical scenarios where similar phenomena could take place. Semiconductor lasers subject to optical feedback produce a rich dynamical behavior, including regimes with important similarities with neuronal dynamics. One of their most interesting ones is the Low Frequency Fluctuation regime (LFF) in which the output power of the lasers suffer sudden dropouts to almost zero power at irregular time intervals when biased close to threshold [64]. It was shown [65, 66] that lasers subject to optical feedback and biased close to threshold are able to operate in an excitable regime, before the onset of the LFFs. This means that a laser prepared in such state is stable under small periodic perturbation of bias current and exhibits the three ingredients of any excitable system, namely: the existence of a threshold for the perturbation amplitude above which the dropout events can occur; the form and size of the dropout events are invariant to changes in the magnitude of the perturbation; a refractory time exists: if a second perturbation is applied at a time shorter than the refractory time, the system no longer responds.

It has been shown both experimentally [67, 68] and numerically [69, 70] that a laser subject to optical feedback can exhibit stochastic [7] and coherence [71] resonance when biased close to threshold. In what follows results concerning experimental and numerical responses of a semiconductor laser subject to optical feedback, biased close to threshold and modulated by two weak sinusoidal signals are described [63]. Two-frequency forcing of dynamical systems has long been studied [67] with an emphasis being usually placed on quasi-periodic dynamics. In contrast, these results show a resonance at a frequency that is absent in the input signals, i.e., GSR.
The experimental setup consisted of an index-guided AlGaInP semiconductor laser (Roithner RLT6505G), with a nominal wavelength of 658 nm (further details can be found in [63]). The driving signal were composed by the superposition of the two immediate superior harmonics of \( f_0 = 4.5 \text{ MHz} \). Fig.17 shows representative time traces and probability distribution functions (PDF) of dropout events. The left plot of the figure corresponds to experimental data for low, intermediate and high amplitude values of the injected signals. It can be clearly seen that for the intermediate amplitude the dropouts are almost equally spaced at a time interval that corresponds grossly to \( 1/f_0 \) (depicted by the double-headed arrow in the middle panel), a frequency that is not being injected. Thus the laser is detecting the subharmonic frequency in a nonlinear way. To better visualize this fact, the PDFs for a large number of dropouts (approximately 1500) is plotted. For the small amplitude (top-right panel in each side) a peak at a time \( 1/f_0 \) can be observed. Also there were other peaks at longer times which indicate that the system responds sometimes to \( f_0 \) although at some others times dropouts are skipped. For the optimum value of the amplitude (middle-right panel in each side) the PDF has a clear peak at \( 1/f_0 \) indicating that the system is resonating with this frequency. For the higher amplitude (bottom-right panel in each side), there are several peaks at different times corresponding to higher frequencies.

Fig.18 shows the results of dropouts statistic when both tones of input frequencies are linearly shifted the same quantity \( \Delta f \). Experiments revealed how the resonant frequency of the laser followed the dynamics predicted by Eq.2, supporting the robustness of the proposed mechanism.

Similar results were found in [72] for two coupled lasers driven separately by a distinct external
perturbation each, and show that the joint system can resonate at a third frequency different from those of the input signals. In other words, the GSR in this case was mediated by the coupling between the dynamical elements. Even though these experiments were performed in the excitable regime of the semiconductor lasers, similar results were found when studying the polarization response of a vertical-cavity surface-emitting laser, driven simultaneously by two (or more) weak periodic signals in the bistable regime [77], confirming the occurrence of GSR.

4.3. Electronic Circuits

GSR was also explored in electronic circuits whose dynamical behavior emulate neuronal dynamics. In what follows, two different configurations using Monostable Schmitt Trigger and Chua circuits are analyzed.

4.3.1. Monostable Schmitt Trigger.

In this section, the behavior of a neuronal-like electronic circuit is experimentally explored when it is stimulated with a complex signal plus noise [73]. The non-dynamical threshold device [74], which compares a complex signal $S_c$ with a fixed threshold, emits a "spike" (i.e. a rectangular pulse of relatively short fixed duration) when it is crossed from below. This behavior emulates, in a very simplified way, the neuronal "firing". The complex signal $S_c$ is formed by adding pure tones with frequencies $f_1 = kf_0$, $f_2 = (k+1)f_0$, ..., $f_n = (k+n−1)f_0$ plus a zero mean Gaussian distributed white noise term.

The circuit implementing the threshold device is comprised by two monostable Schmitt Triggers. The input signal is amplified by the operational amplifier and fed to the first monostable which will trigger or not depending on the amplitude of $S_c$. When it triggers, a pulse is generated in the first monostable during a period of $T_1$ (emulating a neuronal spike). The falling edge of $T_1$ triggers the second monostable. The complemented output of the second monostable is used to clear the first monostable inhibiting further triggering until the expiration of $T_2$ (this emulates the neuronal refractory period). The circuit can be triggered again after the completion of $T_1$ and $T_2$, which are fixed times. The noise intensity was increased in small steps, and held at each step for a fixed time interval. The steps were long enough to collect good statistics, even for low noise intensity levels where rate of spikes is lower. Up to five input frequencies combinations (i.e. $n \leq 5$) were explored.

The output of the circuit was digitized and processed offline to compute intervals of time between triggering, from which inter-spike intervals (ISI) histogram was calculated. The signal to noise ratio (SNR) was computed as before: The ratio between the number of spikes with ISI equal to (or near within ±5%) the time scale of $1/f_0$, $1/f_1$ and $1/f_2$, and the total number of ISI (i.e. at all other intervals).

Fig.19 shows the results from the experiments using harmonic signals composed up to five periodic terms (i.e. $S_c$ with $n = 2, 3, 4, 5$ and $f_0 = 200$ Hz). The amplitude of the deterministic terms are set at 90% triggering level, i.e. there is no triggering in absence of noise. Depicted are the SNR of spikes spaced by intervals close to the periods of the terms comprising the driving signal as well as with $1/f_0$ for increasing noise intensity.

Even though the output was rather incoherent with any of the input frequencies (empty circles and stars), it could be noticed that it was maximally coherent, at some optimal amount of noise, with the period close to $1/f_0$ (filled circles). As in previous cases, $f_0$ was a frequency absent in the signals used to drive the system, showing an experimental manifestation of GSR.

As was shown in numerical simulations, the effects of frequency shift in the harmonic inputs of $S_c$ were also explored in this circuit. With this goal, each harmonic frequency of the input was shift by a $\Delta f$ term. The response of the circuit is plotted in Fig.20. As it was observed in previous sections, the agreement between experimental results and theoretical predictions is remarkable.
Figure 19. GSR in monostable Schmitt Trigger. Signal-to-noise ratios versus noise intensity for signals with two to five frequencies (panels A to D respectively). It is computed as the probability of observing an inter-spike interval close to the time scales (with a 5% tolerance) of the frequencies $f_0$ (filled circles), $f_1$ (empty circles) and $f_2$ (stars). Notice that the largest resonance is always for the ghost $f_0$, while the others are negligible. Reproduced from ref.[73].

Figure 20. Frequency shift experiments in Monostable Schmitt Trigger. Main resonances for signals with two to five frequencies (panels A to D respectively). In each panel, the intervals (plotted as its inverse, $f_r$) between triggered pulses are plotted as a function of $f_1$, which was varied in steps of 40Hz. Family of over-imposed lines are the theoretical expectation (i.e. Eq. 2 with $N = 2, 3, 4, 5$ in panels A through D respectively) for increasing $k = 2 − 7$. Reproduced from ref.[73].
4.3.2. **Pulsed coupled excitable “Chua” circuits**

In Section 3.1, simulations for a binaural configuration of GSR in numerical simulations were discussed. Here, results where the same mechanism is explored experimentally via pulsed coupled electronic neurons [78] are shown. To that end, two excitable electronic circuits were driven by different sinusoidal signals producing periodic spike trains at their corresponding frequencies. Their outputs plus noise were sent to a third circuit that processed these spikes signals.

The model is the electronic implementation of the so-called Chua circuit [79] in the excitable regime. The two input circuits were harmonically driven at two different frequencies, \( f_1 \) and \( f_2 \), generated by a wave generator. The amplitudes of both signals were set above the threshold of the excitable circuits in order to produce periodic spiking at their outputs. These spikes were then fed to a third processing circuit via a voltage follower (which guarantees unidirectional coupling) and an electronic adder. The later also received a noisy signal with bandwidth of 80MHz produced by the wave generator. The two harmonic inputs had frequencies \( f_1 = k f_0 + \Delta f \) and \( f_2 = (k + 1)f_0 + \Delta f \) as explored already along these notes.

The frequencies of the input signals were \( f_1 = 1600\text{Hz} \), \( f_2 = 2400\text{Hz} \) and \( \Delta f = 0 \) was taken for the harmonic case. In this case, both signals were the second and third superior harmonic of \( f_0 = 800\text{Hz} \). The peaks values of signal received for the processing circuit was just below threshold, so it could not fire in absence of noise. Left columns of Fig.21 shows the time series of the output signal for three different values of noise amplitudes: low, optimal and high. The corresponding histograms for the inter-spike intervals were displayed at the right of the time series. Middle panel of Fig.21 shows the coefficient of variation (CV) as a function of the mean value of inter-spike time intervals. Both panels show that there is an optimal amount of noise for which the systems responds at the missing fundamental frequency of the inputs, i.e., GSR.

![Figure 21](image-url)

**Figure 21.** GSR in a binaural configuration of electronic neurons. Left Panels: Influence of the noise intensity in the spiking behavior of the system. The left-column plots show time series for: (a) Low , (b) Optimal, and (c) High values of noise . Plots (d)-(f) are the corresponding probability distribution functions of the interval between spikes. Intermediate values of noise intensity ((b), (e)) show an entrainment of the system at the ghost frequency \( f_0 = 800\text{Hz} \) \( (T_r = 1/f_r = 1.25 \text{ ms}) \). Middle Panel: Coefficient of Variation (CV) of the inter-spike interval vs its mean. The different measurements correspond to increasing values of noise. The minimum of the CV corresponds to the entrainment of the system at the ghost period \( (T_r = 1/f_r = 1.25 \text{ ms}) \). Right panel: Mean spike frequency of the processing circuit for varying frequency shift (\( \Delta f \)). The dashed line corresponds to the theoretical value of \( f_r \) given by Eq. 2 for \( k=2 \) and \( N=2 \). Reproduced from [78].

As in previous work, the influence of detuning the input frequencies was further explored in these experiments. It means that the differences between \( f_1 \) and \( f_2 \) was fixed in \( f_0 \) but they were not longer superior harmonic of \( f_0 \), given that \( f_1 = k f_0 + \Delta f \) and \( f_2 = (k + 1)f_0 + \Delta f \). Right panel of Fig.21 shows two interesting results: one is that experimental data follows theoretical prediction for \( k = 2 \) and \( N = 2 \); Second: there is not ambiguity in resonant frequencies as was reported in previous section. The reason behind this difference is related to the pulsed coupling between input and processing electronic neurons. This was a process of coincidence detection, as happened in simulations shown in Section 3.1, but with the difference that the time window in which the coincidence take places was much shorter in this case, avoiding ambiguity in output responses (See details in [78]).
4.4. Beyond Ghost Stochastic Resonance

Along these notes we have reviewed how a nonlinear system responds to a combination of pure tones with different frequencies. In all the exposed cases, the input frequencies followed a particular relation: the difference between them was always constant and equal to $f_0$, which in the harmonic case corresponds to the missing fundamental. In all cases, the analyzed systems responded with a preferred frequency which was absent in the input.

The dynamics of nonlinear devices, when it is stimulated by more than one frequency plus noise was also studied also from related perspectives. For example, the response of a discrete model system to bichromatic input in the regimes of stochastic and vibrational resonance was numerically analyzed in [80]. The transition between stochastic resonance regime (where frequencies are of the same order) to the vibrational resonance one (where one of the frequencies is much higher that the other) was studied as well as their dependence in bistable and threshold devices. Similar analysis were carried out experimentally in bistable Schmitt Trigger circuits [81], where the authors analyzed the phenomena of mean switching frequency locking and stochastic synchronization and their dependence with input parameters.

Another perspective in the study of the dynamic of nonlinear devices driven by more than one frequency was carried out in [82], where the problem of transport in a noisy environment was studied. In this work, ratchet devices were stimulated by two periodic signals with frequencies $f_1$ and $f_2$ following rational ratios (i.e. $f_1/f_2 = m/n$). The results focused in how the rectification of a primary signal by a ratchet could be controlled more effectively if it is applied a secondary signal with tunable frequency and phase.

5. Summary

When nonlinear systems are driven by noise and a complex stimuli composed by several frequencies, their output exhibits a response characterized by the emergence of a “ghost” frequency which is absent in the input. This phenomena, called Ghost Stochastic Resonance, was proposed to explain a classic paradox in acoustic psychophysics: the missing fundamental illusion. It was later found to provide a theoretical framework to understand a wide variety of problems, from the perception of pitch in complex sounds or visual stimuli, to the explanation of climate cycles. The theory shows that the robustness of this phenomena relies in two simple ingredients present in all the systems analyzed: a linear interference of the periodic inputs and a nonlinear detection of the largest constructive interferences, involving a noisy threshold. Theoretical analysis showed that when the input frequencies are higher harmonics of a given missing fundamental $f_0$, the predominant output frequency is $f_0$. On the other hand, when input frequencies are spaced by $f_0$ but they are no longer superior harmonic of $f_0$, the predominant response frequencies should follow a family linear functions described by Eq.2. The remarkable agreement between theory, simulations and experiments found in the reported results is parameter independent and applies to a wide variety of problems ranging from neurons, semiconductor lasers, electronic circuits to models of glacial climate cycles, as we have reviewed here. Probably there are a equally large number of problems which can be mapped to the same mechanism awaiting to be uncovered.

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