Bifurcations in a simple hydraulic oscillator: the 'Tantalus' cup'

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Abstract. The dynamics of a simple hydraulic oscillator (the 'Tantalus' cup') is studied. Phase-resetting for a single perturbation and locking phenomena in the periodically forced regime are described. A simplistic time-discrete model of the system is proposed. A few other didactic problems are suggested.

Résumé. Le comportement dynamique d'un oscillateur hydraulique simple (le vase de Tantale) a été étudié. Les déphasages induits par l'application de perturbations isolées sont décris, de même que les différents regimes d'entraînement associés à l'application de stimulations périodiques. Un modèle simplifié, où le temps est représenté par une variable à valeur discrète, est proposé. Quelques problêmes sont suggérés à titre d'extension pédagogique.

1. Introduction

Complicated behaviour in deterministic systems has been extensively discussed in the recent past. Simple models of dynamical systems are very attractive because, while preserving the generic properties of the system, they are analytically tractable and allow for a clear understanding of the mechanisms responsible for the dynamics. This paper deals with the dynamics of a non-linear hydraulic oscillator (the 'Tantalus' cup') that exhibits periodic oscillations and more complicated behaviour when periodically perturbed.

The basic apparatus and the main details of the unperturbed oscillation are presented in section 2. The phase resetting behaviour in response to single perturbations and the dynamics resulting from periodic forcing are discussed in section 3. We study, in section 4, a simple time-discrete model describing the main phenomena. In section 5, some didactic experiments and related problems are suggested.

2. Basic apparatus

This oscillator has been known for many years. The reader interested in historical details may consult [1–5]. The basic device is drawn in figure 1 (part A). Water flows at a constant rate μ_i through tube I, increasing the tank water level h_t up to a critical value H' at which the siphon (S) is activated. Typically μ_o, the flow through the siphon, removes water, decreasing h_t. When h_t reaches the level H'', air passing through the siphon interrupts its flow, and a new cycle starts. Part B of figure 1 shows typical oscillations of h_t obtained in a device whose dimensions and parameters are indicated in the legend. h_t is directly proportional to the pressure at the bottom of the tank which can be measured with a transducer. It is seen that the rising phase of the oscillation is fairly linear except at intervals near the activation and deactivation of the siphon. The emptying rate is a non-linear function of the water level in the tank, measured in reference to the output end of the siphon. It slows down as the water level decreases and comes closer to H''. At this point, the siphon is deactivated, and the cycle starts again.

The period of the oscillation as a function of μ_i is shown in figure 2. The non-monotonicity in the function comes from the action of μ_i on both the filling and emptying phase. As μ_i increases, the time needed to fill the tank decreases, explaining the early drop in the period. However μ_i also prolongs the emptying phase, adding additional water to the tank while the siphon is acting to remove it. At some value of μ_i the prolongation of the emptying phase starts to exceed the corresponding abbreviation of the filling phase, and the period begins to increase. Finally, a value of μ_i is reached (near point U in figure 2) beyond which there is always a level of water such that $\mu_i = \mu_o$. From there, the system always adjusts itself to the level where the siphon removes exactly the liquid provided by μ_i. The height of this stable water level increases with μ_i. Mathematically, a stable equilibrium point has

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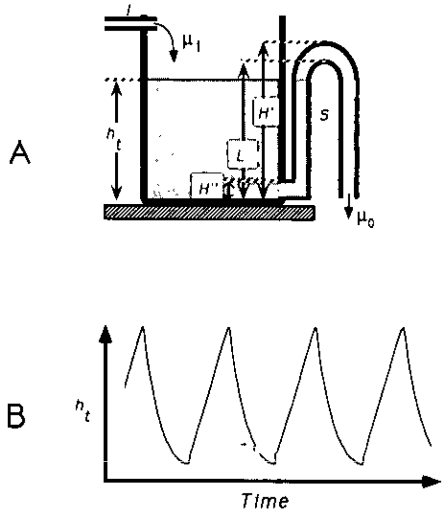


Figure 1. Basic apparatus. Flow (μ_i) is provided through tube I at a constant rate. Oscillation of h_t is constrained, for appropriate μ_i values, to the limit H'–H''. When the water level reaches the height L, water starts leaking through the siphon. When it reaches H', the siphon is fully activated and pumps water out of the tank at a rate μ_0 which is a function of h_t. B: typical oscillations of h_t (Period = 85 s). Device dimensions and parameters here as well as in figures 2 to 4 are: tank diameter: 10 cm, S diameter: 1 cm, H'–H'': 9.5 cm and μ_i: 11 $\mathrm{cm}^3\,\mathrm{s}^{-1}$.

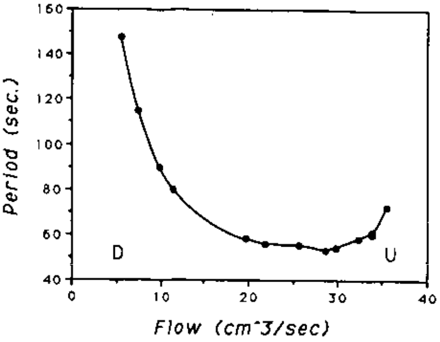


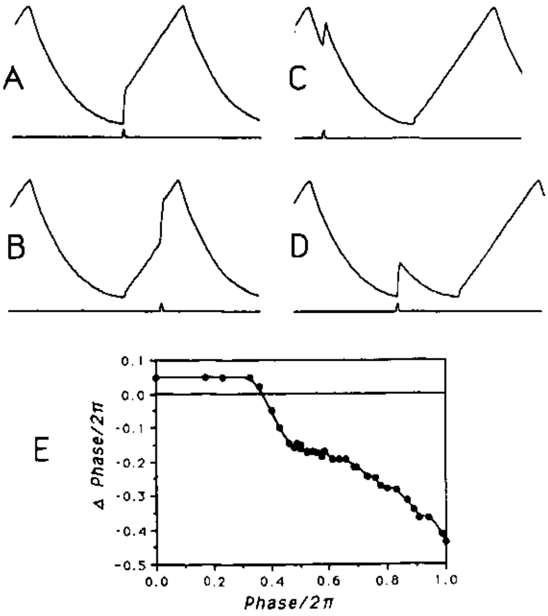
Figure 2. Oscillation period as a function of flow rate. For μ_i (flow) below D, the leaking through the syphon stops the oscillation and the water level is maintained between the points L and H' of figure 1A. For μ_i beyond U, the oscillations also stop. The system is then locked in the emptying phase, at water height such that $\mu_i = \mu_0$.

appeared, and the system no longer oscillates. For a different reason, a stable equilibrium point also exists at very low μ_i (near the point D in figure 2). In this case, the water level reaches the level L in figure 1, water then leaks through the siphon, but μ_i is too low to fill the curved upper section of the siphon and activate it. The water level stays constant, unless water is added to start the siphon. Under this condition the device can be considered as an excitable non-autonomous system.

3. Phase resetting and periodic forcing

The oscillation of the system can be externally perturbed by increasing the flow rate for a relatively brief interval of the spontaneous cycle. This can be done by an electromagnetic valve connected to a second reservoir and opened by short current pulses provided by a timing device. As shown in figure 3, the effect of the perturbation changes with the phase of the cycle where it occurs. Taking as a reference point the moment where the water level reaches its minimum, any point of the cycle can be expressed as a fraction of the period. When the perturbation is applied while h_t is growing, the cycle is abbreviated (parts A and B of figure 3), and the system returns sooner to its minimum. This abbreviation, expressed as a fraction of the natural period, can be interpreted as a phase shift. The perturbation is then seen as advancing the phase of the system. The evolution of the system is linear when h_t is growing. Therefore, for most of the growing phase the same perturbation will advance the next oscillation by the same amount, irrespective of the phase at which it is applied. However, for perturbations applied near the top of the oscillation, that exceed H' (see figure 1A), there is a mixed effect. The stimulus shortens the cycle by increasing h_t and activating the siphon, while the

Figure 3. Phase shift induced by a single perturbation (flow rate = 208 $\mathrm{cm}^3\,\mathrm{s}^{-1}$ during 1 s) applied at different phases of the spontaneous cycle described in figure 1B. Parts A and B, perturbations at the rising phase. Parts C and D, perturbations at the decaying phase. Part E: phase resetting curve.



fluid over H' will act to delay the next oscillation. As the perturbation comes closer to the top of the cycle the former effect becomes more important and changes the sign of the phase shift. Finally there is a progressive delay as the forcing is applied later in the emptying phase (see parts C–D of figure 3). This happens since h_t evolves in a non-linear fashion during the emptying phase, the decay of the water level being much slower as the system moves closed to H''. The fixed amount of water added by the perturbation will take longer to be removed by the siphon if it is added near the end of the cycle, and the cycle will be more prolonged. This prolongation can be interpreted as a backward phase shift of the system by the perturbation. Part E of figure 3 shows the experimental phase response curve determined for pulses of 1 second at a flow of 208 $\mathrm{cm^3\,s^{-1}}$. As predicted earlier, the curve shows a constant positive phase shift for early pulses which falls in the rising part of the h_t cycle and a phase-dependent negative shift for pulses falling in the decaying part of the cycle.

The response of the oscillator to periodic perturbations is now investigated with pulses of 1 s duration (same amplitude as in figure 3) at several periods. In figure 4 are shown some of the stable patterns obtained as the forcing period is decreased from 100 to 14 s when the spontaneous cycle is 85 s. Each stable entrainment pattern can be associated with a rotation number [6–8] defined as m/n; where m is the number of full oscillations and n the number of pulses applied during the periodic sequence.

The occurrence of stable phase-locking patterns can be understood using the phase resetting curve. Consider, as an example, the trace of figure 4 where a stable 1 : 1 pattern is detected for a period of stimulation ($\tau_s = 100$ s) longer than the natural period of the system ($T = 85$ s). In this case, the phase of the system (ϕ) must be the same at the beginning of any stimulus, having covered exactly one full cycle in the time between two successive stimuli (which is normalized and can be expressed as τ_s/T). This will happen if the perturbation induces a phase shift ($\Delta\phi$) such that

$$\phi + \Delta\phi + \tau_s/T = \phi + 1 \qquad (1a)$$

i.e.,

$$\Delta\phi = 1 - \tau_s/T < 0. \qquad (1b)$$

Phase delays (i.e. $\Delta\phi < 0$) occur for perturbations in the emptying phase of the cycle. If a phase exists such that its associated $\Delta\phi$ satisfies equation (1b), then, after some initial transients the system will adjust to this phase for each stimulus, giving rise to a 1 : 1 pattern as in the case of figure 4 (trace at the top left). The same analysis holds for the 1 : 1 phase locking seen at $\tau_s = 75$ s (second left panel of figure 4). In this case a stable 1 : 1 pattern occurs for stimulations in the rising phase, since τ_s is shorter than T, consequently $\Delta\phi$ must be greater than 0 (see equation (1b)). The value of $\Delta\phi$ necessary to maintain a 1 : 1 pattern increases as τ_s is further decreased. At some rate (e.g., $\tau_s = 55$ s), for a given amplitude, the specific stimulation is no longer able to produce a phase shift large enough to maintain 1 : 1 phase-locking. A new stable pattern must emerge, where the same phase recurs (the system will then have completed m full cycles), after a fixed number of stimulations n (yielding an $m : n$ phase-lock, see figure 4). Since $\tau_s < T$, then $n < m$ (i.e., n $\tau_s = m\,T$), and phase delay will alternate with phase advance. The location of the transitions is dependent on the maximum value of $\Delta\phi$, and changes with the amplitude and duration of the stimulation. It is seen in figure 4 that the rotation number decreases monotonically as the forcing period is shortened. The sequence of $n : m$ phase-locking is predictable, following the so-called Farey rule [6–8]. Finally, there is a small parameter region where aperiodic activity occurs, typically as occasional intermissions of one or two cycles of a neighbouring phase-locking pattern (in the Farey tree).

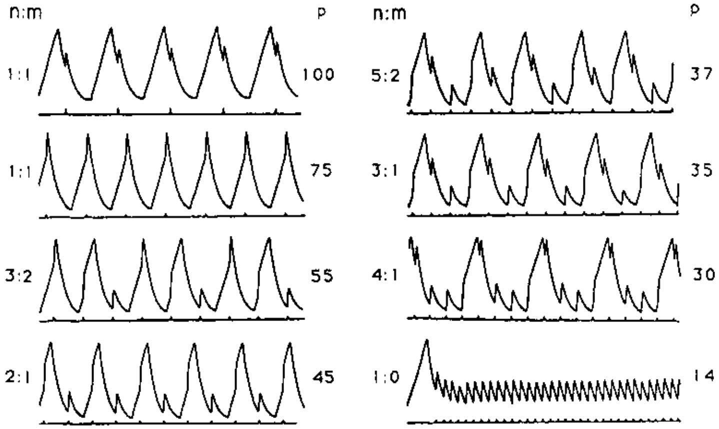


Figure 4. Examples of phase-locking using periodic pulses (208 $\mathrm{cm^3\,s^{-1}}$, duration 1 s) at several periods ("P", indicated in s at right of each trace). Fraction at left of each time series indicates the stable $n : m$ pattern.

4. The finite-difference model

Except for the interval between the beginning of the leaking phase and the full activation of the siphon, it is relatively easy, using Bernoulli's law, to write down the differential equations governing the evolution of the water tank level. However, these equations are highly sensitive to all the parameters of the system (radius of the tank and of the sink, water flux, etc). Our purpose is to obtain a simple iterated function [9] that will reproduce the important aspects of the dynamics and will still be applicable to a wide class of oscillators, irrespective of their detailed physical description. In particular, we want to obtain the organization of the successive phase-locking patterns allowed by the system.

Consider first $H' = 1$ and $h_t = 0$ at H'' (see figure 1), and let the variation of h_t be confined to the interval [0, 1]. Consider also that h_t is recorded only at fixed time intervals, such that all the dynamics is reduced to a sequence of value $h_{t(i)}$, $i = 1, n$. From the arguments in section 2, h_t increases linearly during the phase, so that we can write: if $h_{t(i)} - h_{t(i-1)} > 0$

$$h_{t(i+1)} = f\,(h_{t(i)}) \qquad \forall i / 0 \leqslant h_{t(i)} \leqslant 1$$

where f is:

$$\begin{aligned} h_{t(i+1)} &= a + h_{t(i)} && \text{if } a + h_{t(i)} < 1 \\ &= 1 && \text{if } a + h_{t(i)} \geqslant 1 \end{aligned} \qquad (2a)$$

where a is the normalized amount of water added during the time step i.

In the decaying phase, we also write: if $h_{t(i)} - h_{t(i-1)} \leqslant 0$, and $h_{t(i)} > h_{\min}$,

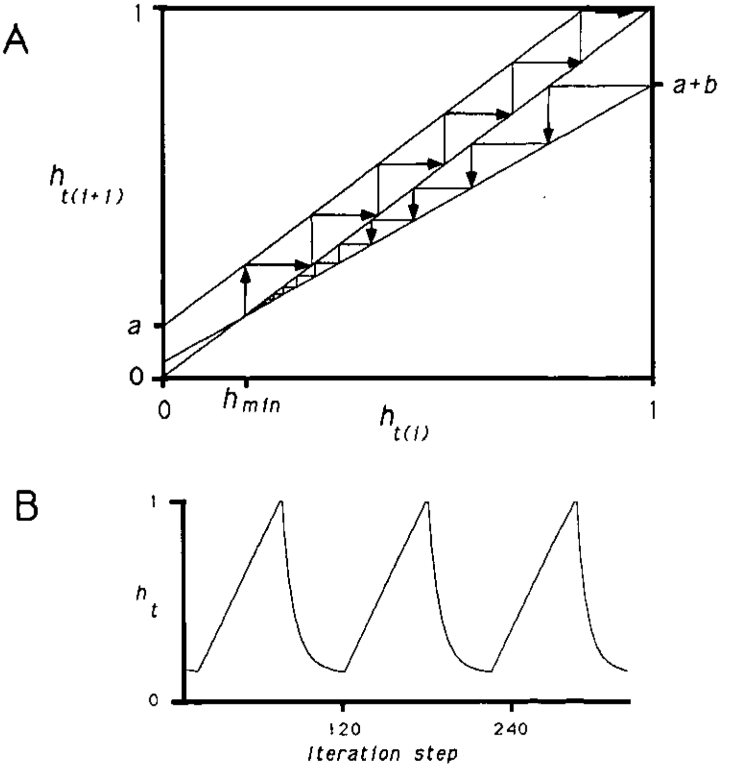
$$h_{t(i+1)} = g\ (h_{t(i)})$$

Even though g has a more complex formula, we write for simplicity:

$$\begin{aligned} h_{t(i+1)} &= a + bh_{t(i)} && \text{if } a + bh_{t(i)} \leqslant 1 \\ &= 1 && \text{if } a + bh_{t(i)} > 1 \end{aligned} \qquad (2b)$$

where $h_{\min}$ is the minimum level below which the siphon is deactivated ($h_{\min} = H''$); b is a constant ($0 < b < 1$) representing the emptying effect of μ_o and is related to the dimensions of the tank and of the siphon. In both the filling and emptying phase $a = a_c$, if there is no stimulation, with a_c being the fixed amount of water provided by μ_i during one time step. During a stimulation, $a = a_c + a_s$, where a_s is the

Figure 5. A: schematic of the iteration of one cycle and parameters of the map of equation (2). B: cycle obtained with $a_c = 0.0144$, $b = 0.9$ and $h_{\min} = 0.15$. The period of the cycle is 109 iteration steps.



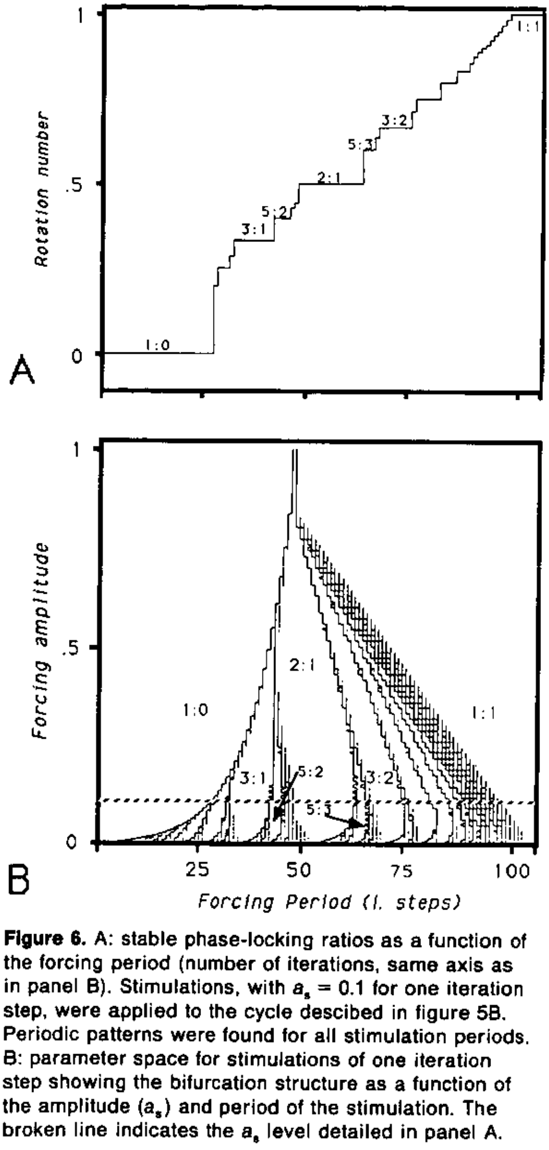


Figure 6. A: stable phase-locking ratios as a function of the forcing period (number of iterations, same axis as in panel B). Stimulations, with $a_s = 0.1$ for one iteration step, were applied to the cycle descibed in figure 5B. Periodic patterns were found for all stimulation periods. B: parameter space for stimulations of one iteration step showing the bifurcation structure as a function of the amplitude (a_s) and period of the stimulation. The broken line indicates the a_s level detailed in panel A.

amount of water provided by the stimulation. In the autonomous regime, the system is characterized by the three parameters a_c, b and $h_{\min}$ which together determine the nature of the system's unique stable state. The emptying branch (equation (2*a*)) has a stable fixed point; i.e., $h_{t(i)} = h_{t(i+1)} = h_t f$, where:

$$h_t f = a_c/(1-b). \qquad (3)$$

If $h_{\min} \leqslant h_t f$, there will be no oscillation, and the system will stay trapped at $h_t f$ during the emptying phase. If $h_{\min} > h_t f$, the system cannot reach the equilibrium and will oscillate. Equation (2) can be solved graphically. Part A of figure 5 is the map of equation (2) showing the iteration of a typical cycle, while part B shows a time series for the specific set of parameters described in the legend. The branch at the left side of the identity line is the solution of $f(h_t)$, representing the filling phase, and that at the right side the solution of $g(h_t)$, describing the emptying phase.

Repetitive stimulations of various amplitudes and periods were applied to the cycle depicted in figure 5B, and results are presented in figure 6. In part A, stimulations with $a_s = 0.1$ for one iteration step were applied with period ranging from 1 to 100 iteration steps. For each period, a stable phase-locking pattern was detected, and the resulting rotation numbers are plotted as a function of the stimulation period. As for the experimental results of figure 4, there is a monotonic decrease of the rotation number as the period is abbreviated. Each phase-locking pattern extends over an interval of stimulation period, and the curve takes the general form of a staircase (the so-called Devil's staircase' [6]) following the prediction of Farey's arithmetic. Figure 6B presents the complete bifurcation structure for a_s ranging from 0.001 to 1.000. For each forcing amplitude, the two stimulation periods setting the limit of the period interval for each stable rotation number (end points of each step in the staircase of panel A) are plotted. As discussed in the previous section, it is seen that the bifurcation structure changes with the amplitude of stimulation. As the amplitude increases, the number and period extension of the phase-locking patterns between 1 : 1 and 1 : 0 entrainments diminish, being finally reduced to the three patterns 1 : 1, 2 : 1 and 1 : 0. This occurs when a_s becomes $\geqslant 1 - h_{\min}$. From this stimulation amplitude, h_t is always equal to 1 at the end of a stimulation, irrespective of the phase at which the perturbation was applied. The phase is then $1 + \tau_s/T$ at the beginning of the next stimulation. If τ_s is greater than the emptying time (48 iteration steps, for the free running cycle of figure 5B), the following stimulation comes in the filling phase, fills the tanks and starts the syphon, resulting in a 1 : 1 phase-locking pattern. If τ_s is shorter than the emptying time, the stimulation falls always in the decaying phase, giving a 1 : 0 pattern. The 2 : 1 patterns existing for $\tau_s = 48$ steps appear because equation (2*a*) states that, at the end of the filling phase, the system must stay at $h_t = 1$ for two iteration steps before the beginning of emptying.

In spite of its gross simplifications (i.e., there is no leaking phase, the water level is not allowed to go over H'), the finite-difference model gives a global bifurcation structure (figure 6) that is very similar to the one measured experimentally in the system, as well as to what is predicted by the set of differential equations which describe it [10]. Furthermore, results similar to those in figure 6 have also been observed for biological oscillators forced periodically by external low amplitude pulses [6]. The minimal model of equation (2) is in some sense generic and embeds the fundamental dynamical properties of a variety of natural systems [11].

5. Conclusion (some additional questions and didactic experiments)

There are additional questions and experiments which could be addressed using this model. The following are just some which it might be interesting to pursue.

(1) The form of $g(h_t)$ is obviously not the exact description of the decaying phase of the oscillation. Is there a better representation? Will there be relevant differences in the dynamics predicted using a different formulation?

(2) If the incoming flow is connected to a point near the bottom of the tank and μ_i is supplied at a constant pressure rather than at a constant flow: how does equation (2) change?; how would the resetting behaviour change (figure 3)?; and what would be the changes in the dynamics during periodic forcing?

(3) If the forcing is instrumented such that the perturbations are now periodic changes in the siphon height (i.e., changes in H', see figure 1): are there are new classes of dynamics possible in the system?

(4) Here we have just discussed the case of a cylindrical tank with constant diameter. It would be interesting to investigate other tank geometries. Would it be possible to obtain a sinusoidal oscillation?

(5) Interaction of two oscillators can be demonstrated by coupling two devices, connecting the main tank through a tube whose diameter can be visualized as the coupling constant. Non-linear or linear diffusive coupling can be instrumented by properly selecting the height at which both tanks are interconnected.

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